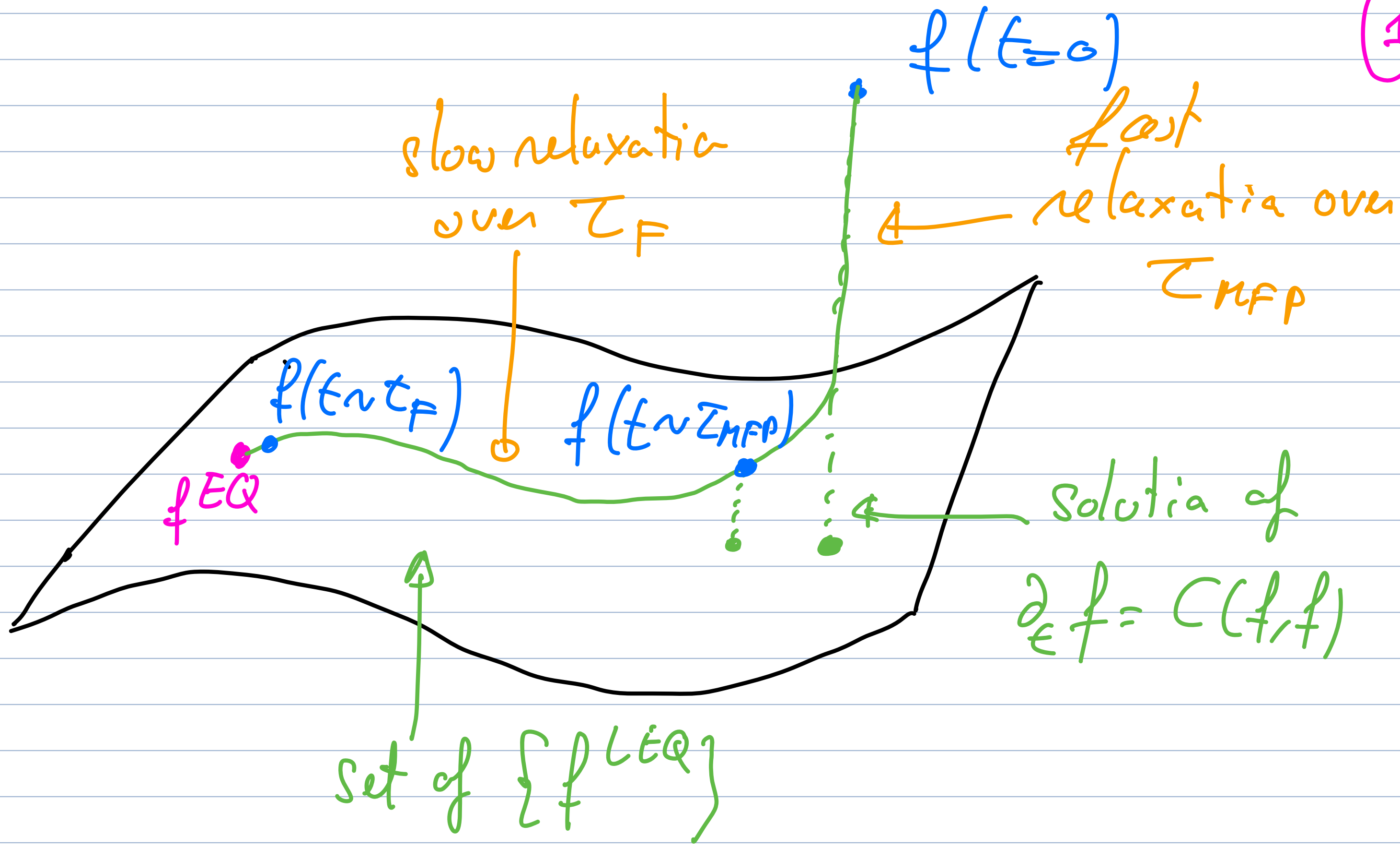


BE: $\partial_{\hat t} f + \underbrace{\frac{\partial f}{\partial \vec q} \cdot \frac{\vec p}{M}}_{\equiv \frac{1}{\tau_F} L_F f} = \underbrace{C(f,f)}_{\equiv \frac{1}{\tau_{MFP}} \hat C(f,f)}$; $t = \hat t\, \tau_F$; $\varepsilon = \frac{\tau_{MFP}}{\tau_F}$

$$f(\vec q, \vec p, t) = f_0(\vec q, \vec p, t) + \varepsilon f_1(\vec q, \vec p, t) + \mathcal{O}(\varepsilon^2)$$

Order by order:

$\mathcal{O}(\frac{1}{\varepsilon})$ $\quad 0 = C(f_0, f_0) \Rightarrow f_0 = f^{LEQ}$

$\mathcal{O}(1)$ $\quad \partial_{\hat t} f_0 + L_1 f_0 = \hat C(f_0, f_1) + \hat C(f_1, f_0) \simeq -f_1$

leading order

$$f_0(\vec q, \vec p, t) = f^{LEQ}(\vec p, \vec q) = \tilde\gamma(q,t)\, e^{-\vec\alpha(\vec q,t)\cdot \vec p - \beta(\vec q,t)\left[\frac{\vec p^{\,2}}{2m} + U(\vec q)\right]}$$

Characterized by

density field $n(\vec{q},t) = \int d\vec{p}\, f(\vec{q},\vec{p},t)$

velocity field $\vec{u}(\vec{q},t) = \frac{1}{n}\int d\vec{p}\, f(\vec{q},\vec{p},t)\,\vec{v} \equiv \langle \vec{v} \rangle_{f(\vec{q},\vec{p},t)}$

energy fields / temperature fields

$$\begin{cases} \varepsilon = \langle \frac{1}{2} m\, \delta\vec{v}^2 \rangle = \frac{1}{2} m\left(\langle \vec{v}^2\rangle - \vec{u}^2\right) \\ T(\vec{q},t) = \frac{2}{3}\frac{\varepsilon(\vec{q},t)}{k_B} \text{ such that } \varepsilon = \frac{3}{2} k_B T \end{cases}$$

Evolution of the slow fields

$$\boxed{D_t n \equiv (\partial_t + u_\alpha \partial_{q_\alpha}) n = - n\, \partial_{q_\alpha} u_\alpha}$$

$$\boxed{M\, D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta} \cdot P_{\alpha\beta} \Leftrightarrow M\, D_t \vec{u} = -\frac{1}{n}\vec{\nabla}\cdot \bar{\bar{P}}}$$

with $P_{\alpha\beta} = M n \langle (u_\alpha - v_\alpha)(u_\beta - v_\beta)\rangle = M n \langle \delta v_\alpha\, \delta v_\beta \rangle$

$$\boxed{\partial_t T + u_\alpha \partial_\alpha T = -\frac{2}{3 n k_B}\partial_\alpha h_\alpha - \frac{2}{3 n k_B} P_{\alpha\beta} u_{\alpha\beta}}$$

where

- $u_{\alpha\beta} = \frac{1}{2}\left(\partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha\right)$ is called the strain rate tensor
- $h_\alpha = \frac{n M}{2}\langle \delta v_\alpha\, \delta v_\beta \delta v_\beta \rangle$ is the kinetic energy flux along $\vec{u}_\alpha$, a.k.a. heat flux

Closure: To compute the evolution of $n, T, \vec{u}$, we need to compute $P_{\alpha\beta} = nM \langle \delta v_\alpha \delta v_\beta \rangle$ & $h_\alpha = \frac{nM}{2} \langle \delta v_\alpha \delta v_\beta \delta v_\beta \rangle$, which we can do perturbatively using f:

$$f_0 \Rightarrow P^0_{\alpha\beta}, h^0_\alpha \Rightarrow \partial_t (n, T, \vec{u})^0 \Rightarrow f_1 \Rightarrow P^1_{\alpha\beta}, h^1_\alpha \Rightarrow \partial_t (n, T, \vec{u})^1 \text{ etc.}$$

Notation: A_0 or A^0 refers to the order of the approximation

2.4.3) Leading order dynamics

Given $n, T, \vec{u}$, we can determine f_0 through:

$$\left.\begin{aligned}
&\int f_0 \, dp = n(q) \\
&\int \vec{v} f_0 \, dp = n(\vec{q}) \vec{u}(q) \\
&\int \frac{M}{2} (\vec{v} - \vec{u})^2 f_0 \, dp = n(\vec{q}) \, \varepsilon(\vec{q}) = \frac{3}{2} k_B T(\vec{q})
\end{aligned}\right\} \quad f_0(\vec{q}, \vec{p}, t) = \frac{n(\vec{q})}{[2\pi M k_B T]^{3/2}} e^{-\frac{(\vec{p} - M\vec{u})^2}{2MkT}} = \frac{n}{(2\pi M k_B T)^{3/2}} e^{-\frac{M \delta \vec{v}^2}{2kT}}$$

Pressure & heat flux

$$P^0_{\alpha\beta} = n M \langle (v_\alpha - u_\alpha)(v_\beta - u_\beta) \rangle_0 = n M \cdot \delta_{\alpha\beta} \cdot \frac{k_B T}{M} = n k_B T \delta_{\alpha\beta}$$

$\Rightarrow$ Isotropic pressure given by ideal gas law

$h^0_\alpha = 0$ because it involves an odd moment of δv

Leading order dynamics

$$\partial_t n + u_\alpha \partial_{q_\alpha} n = -n\, \partial_{q_\alpha} u_\alpha \qquad (1)$$

$$m\, \partial_t u_\alpha + m\, u_\beta \partial_{q_\beta} u_\alpha = -\frac{1}{n} \partial_{q_\alpha} [n k T] \qquad (2)$$

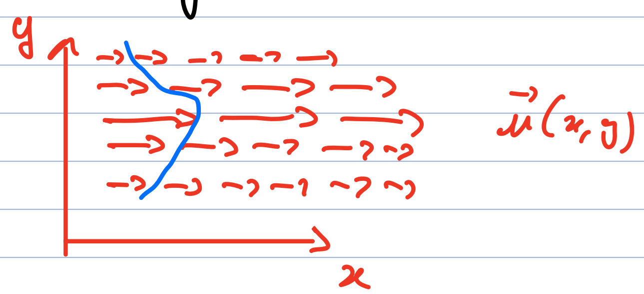
$$\partial_t T + u_\alpha \cdot \partial_\alpha T = -\frac{2}{3} T\, \partial_{q_\alpha} u_\alpha \qquad (3)$$

At order 0, $f = f_0$ is left invariant by collisions but $\partial_t (n^0, u^0, T^0) \neq 0$
$\Rightarrow$ induce some evolution of f_0 already $\Rightarrow$ maybe we are lucky & we get equilibration at leading order.

Does this dynamics relax?

Let's consider an initial condition in which the gas is sheared

$n = n_0$ constant, $T = T_0$ constant, $\vec{u} = u(y)\, \vec{e}_x$



In eq (2), $\partial_{q_\alpha}(n_0 k T_0) = 0$

$$\left.\begin{array}{l} \partial_{q_\alpha}(n_0 k T_0) = 0 \\ u_\beta \partial_{q_\beta} \vec{u}_\alpha = \delta_{\alpha x} \underbrace{u_y}_{=0} \partial_{q_y} \vec{u}_x = 0 \end{array}\right\} \partial_t \vec{u}_x = 0 \Rightarrow \vec{u} \text{ not relaxing to } \vec{u} = \vec{0} \ldots$$

$\Rightarrow f^{LEQ}$ does not relax to f^{EQ} if we compute h & P to leading order :-(

2.4.4) First order hydrodynamics

$$f_1 = -\tau_F\left[\partial_t f_0 + \frac{\vec{p}}{m}\cdot\partial_{\vec{q}} f_0\right]$$ easier than working with $\hat{t}$.

$$f = f_0 + \frac{\tau_{MFP}}{\tau_F} f_1 = f_0\,(1 + g_1)$$

$$-g_1 = \tau_{MFP}\left[\partial_t \ln f_0 + \frac{\vec{p}}{m}\cdot\partial_{\vec{q}}\ln f_0\right] = \tau_{MFP}\left(\partial_t + v_\alpha\partial_{q_\alpha}\right)\left[\ln n - \frac{3}{2}\ln T - m\frac{(\vec{v}-\vec{u})^2}{2k_BT}\right]$$

We need to compute g_1 to leading order in ε

$\Rightarrow$ use 0^{th} order hydrodynamics for the right-hand-side

(i) use $\partial_t + v_\alpha\partial_{q_\alpha} = D_t + (v_\alpha - u_\alpha)\partial_{q_\alpha} = D_t + \delta v_\alpha\partial_{q_\alpha}$

(ii) leading order hydrodynamic equations

$$\begin{cases} D_t n = -n\,\partial_{q_\alpha} u_\alpha \\ m D_t u_\alpha = -\frac{1}{n}\partial_{q_\alpha}\left[n kT\right] \\ D_t T = -\frac{2}{3}T\,\partial_{q_\alpha} u_\alpha \end{cases}$$

$$\Rightarrow \boxed{g_1 = -\frac{\tau_{MFP}\, m}{kT} u_{\alpha\beta}\left(\delta v_\alpha\delta v_\beta - \frac{\delta v^2}{3}\delta_{\alpha\beta}\right) - \frac{\tau_{MFP}}{T}\delta v_\alpha\left(\partial_\alpha T\right)\left(\frac{m\,\delta\vec{v}^2}{2kT} - \frac{5}{2}\right)}$$

Proof:

$\delta_{\alpha\beta}\partial_{q_\alpha}u_\beta = \delta_{\alpha\beta}u_{\alpha\beta}$

$$-\frac{g_1}{\tau_{MFP}} = \underbrace{-\partial_{q_\alpha}u_\alpha - \frac{3}{2}\left(-\frac{2}{3}\partial_{q_\alpha}u_\alpha\right)}_{=0} - \frac{\cancel{m}}{k_BT}\delta v_\alpha\frac{1}{n\cancel{m}}\partial_{q_\alpha}(n kT) - \frac{m}{2k_B}\delta\vec{v}^2\left(+\frac{2}{3T}\partial_{q_\alpha}u_\alpha\right)$$

$$+\delta v_\alpha\frac{\partial_{q_\alpha}n}{n} - \frac{3}{2T}\delta v_\alpha\partial_{q_\alpha}T + \frac{m}{kT}\underbrace{\delta v_\beta\,\delta v_\alpha\,\partial_{q_\alpha}u_\beta}_{\delta v_\beta\,\delta v_\alpha\, u_{\alpha\beta}} - \frac{1}{2}\frac{m}{k_B}\delta\vec{v}^2\left(-\frac{1}{T^2}\delta v_\alpha\partial_{q_\alpha}T\right)$$

$$-\frac{g_1}{\tau_{MFP}} = -\delta v_\alpha \left[\frac{\partial_\alpha n}{n} + \frac{\partial_\alpha T}{T} - \frac{\partial_\alpha n}{n} + \frac{3}{2T}\partial_\alpha T\right] + \frac{M}{kT}\delta v_\alpha \delta v_\beta u_{\alpha\beta} - \frac{M}{3kT}\delta \vec{v}^{\,2}\delta_{\alpha\beta} u_{\alpha\beta} + \frac{M}{2k_B T}\delta\vec{v}^{\,2}\delta v_\alpha \frac{\partial_\alpha T}{T}$$

$$-\frac{g_1}{\tau_{MFP}} = \frac{M}{kT} u_{\alpha\beta}\left(\delta v_\alpha \delta v_\beta - \frac{\delta\vec{v}^{\,2}}{3}\delta_{\alpha\beta}\right) + \frac{\delta v_\alpha \partial_\alpha T}{T}\left(\frac{M}{2k_B T}\delta\vec{v}^{\,2} - \frac{5}{2}\right)$$

(III) Compute h & P to $\mathcal{O}(\varepsilon)$

$$f = f_0(1+g_1) \Rightarrow \langle O \rangle_f = \langle O(1+g_1)\rangle_0$$

$\hookrightarrow f_0$

$$P_{\alpha\beta} = nM\langle \delta v_\alpha \delta v_\beta\rangle = nM\langle \delta v_\alpha \delta v_\beta\rangle_0 - \frac{n\tau_{MFP}M^2}{kT}u_{\gamma\delta}\Big[\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta\rangle_0 - \frac{1}{3}\delta_{\gamma\delta}\langle \delta v_\alpha \delta v_\beta \delta\vec{v}^{\,2}\rangle_0\Big] + 0$$

$\hookrightarrow \langle \underbrace{\delta v \ldots \delta v}_{\text{odd \# of terms}} \rangle$

Wick theorem:

$$\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta\rangle_0 = \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)\underbrace{\langle \delta v_\alpha^2\rangle\langle \delta v_\beta^2\rangle}_{\frac{(k_B T)^2}{M^2}}$$

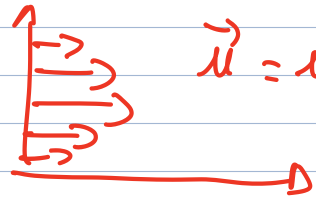
$$u_{\gamma\delta}\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\delta\rangle_0 = \frac{(kT)^2}{M^2}\left[2u_{\alpha\beta} + u_{\gamma\gamma}\delta_{\alpha\beta}\right]$$

$$\langle \delta v_\alpha \delta v_\beta \delta v_\gamma \delta v_\gamma\rangle_0 = \frac{(k_B T)^2}{M^2}\Big[\delta_{\alpha\beta}\underbrace{\delta_{\gamma\gamma}}_{3} + 2\underbrace{\delta_{\alpha\gamma}\delta_{\beta\gamma}}_{\delta_{\alpha\beta}}\Big] = 5\delta_{\alpha\beta}\frac{(k_B T)^2}{M^2}$$

$$\Rightarrow P^{(1)}_{\alpha\beta} = n k_B T\delta_{\alpha\beta} - n k_B T\tau_{MFP}\left[2u_{\alpha\beta} + u_{\gamma\gamma}\delta_{\alpha\beta} - \frac{5}{3}\delta_{\alpha\beta}u_{\gamma\gamma}\right]$$

$$P^{(1)}_{\alpha\beta} = n k_B T\delta_{\alpha\beta} - 2n k_B T\tau_{MFP}u_{\alpha\beta} + \frac{2}{3}n k_B T\delta_{\alpha\beta}\tau_{MFP}\partial_\gamma u_\gamma$$

Relaxation of shear flow:



$\vec{u} = u(y)\,\vec{e}_x$

$$u_{\alpha\beta} = \frac{1}{2}\left[\partial_\alpha u_\beta + \partial_\beta u_\alpha\right] = \frac{1}{2}\left[\delta_{\alpha y}\delta_{\beta x} + \delta_{\beta y}\delta_{\alpha x}\right] u'(y)$$

$$P_{\alpha\beta} = n k_B T\,\delta_{\alpha\beta} - n\,\tau_{MFP}\,kT\left[\delta_{\alpha y}\delta_{\beta x} + \delta_{\alpha x}\delta_{\beta y}\right] u'(y) \Rightarrow \text{not diagonal}$$

$$M D_t u_\alpha = -\frac{1}{n}\,\partial_{q_\beta} P_{\alpha\beta}$$

$$M\partial_t u_x + M\underbrace{u_x \cdot \partial_x u_x}_{=0} + M\underbrace{u_y}_{=0} \cdot \partial_y u_x = -\frac{1}{n}\partial_y\left(-n\,\tau_{MFP}\,kT\,u'(y)\right)$$

$n = n_0$

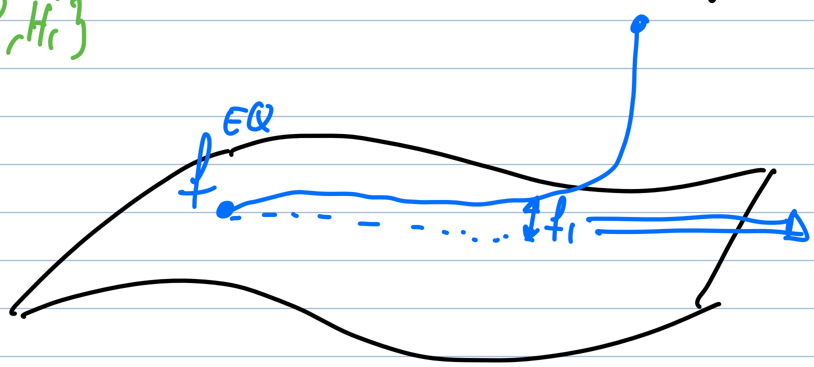
$$\Rightarrow \partial_t u_x = \underbrace{\frac{\tau_{MFP}\,kT}{M}}_{\nu}\,\partial_{yy} u(y)$$

ν is the kinematic viscosity ($\mu = nM\nu$ is the dynamic viscosity).

What drives the relaxation to $\vec{u} = 0$ & equilibrium?

(I) f_1 alters the statistics of f and the values of $P_{\alpha\beta}$, h_α

(II) transport then makes the conserved field relax
$\{f, H_1\}$



$P_{\alpha\beta} = \delta_{\alpha\beta}\,nkT + \delta P$
drives transport

Comment: one can also capture the heat flux $h_\alpha = -K\partial_{q_\alpha} T$

K is the thermal conductivity, $K = \frac{5}{2}\frac{n}{M}\tau_c k_B^2 T$

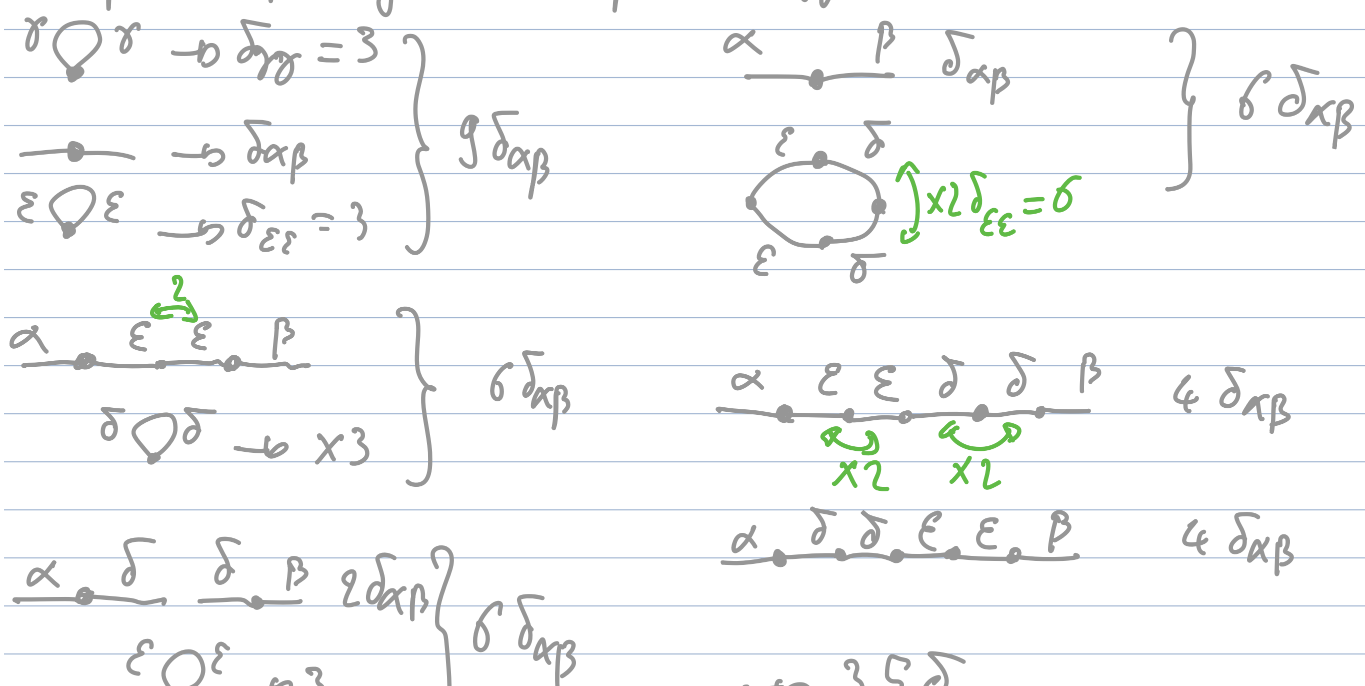
Proof:

$\ast\ h_\alpha = \frac{mM}{2}\left\langle \delta v_\alpha\, \delta \vec{v}^{\,2} (1+g_1)\right\rangle_0$ → only even powers matter
→ — odd — → in g_1

$$h_\alpha = -\frac{mM\tau_{MFD}}{2T}\left(\partial_{q_\beta} T\right)\left\langle \frac{M}{2kT}\left[\delta\vec{v}^{\,2}\right]^2 \delta v_\beta\, \delta v_\alpha - \frac{5}{2}\,\delta \vec{v}^{\,2}\, \delta v_\alpha\, \delta v_\beta \right\rangle$$

$\ast$ We need to compute $\left\langle \delta v_\gamma\, \delta v_\gamma\, \delta v_\varepsilon\, \delta v_\varepsilon\, \delta v_\alpha\, \delta v_\beta \right\rangle = \frac{k^3T^3}{M^3}\,\delta_{\alpha\beta} \times$ (combinatorial factor)

All possible pairings ① Graph strategy



$\Rightarrow 35\,\delta_{\alpha\beta}$

② Math strategy. Sum only if $\sum_i$ explicit.

$$\sum_{\varepsilon,\delta}\left\langle \delta v_\alpha \delta v_\beta\, \delta v_\varepsilon \delta v_\varepsilon\, \delta v_\delta \delta v_\delta \right\rangle = \delta_{\alpha\beta}\left\langle \delta v_\alpha^2 \sum_\varepsilon \delta v_\varepsilon^2 \sum_\delta \delta v_\delta^2 \right\rangle$$

$$= \delta_{\alpha\beta}\Big[\left\langle \delta v_\alpha^6\right\rangle_{\varepsilon=\delta=\alpha} + 2\sum_{\varepsilon\neq\alpha}\left\langle \delta v_\alpha^4\, \delta v_\varepsilon^2\right\rangle_{\delta=\alpha}$$

$$+ \sum_{\varepsilon\neq\alpha}\left\langle \delta v_\alpha^2\, \delta v_\varepsilon^4\right\rangle + \sum_{\varepsilon\neq\delta\neq\alpha}\left\langle \delta v_\alpha^2\, \delta v_\varepsilon^2\, \delta v_\delta^2\right\rangle\Big]$$

$$= \delta_{\alpha\beta}\Big[5\langle\delta v_\alpha^2\rangle\langle\delta v_\alpha^4\rangle + 4\langle\delta v_\alpha^4\rangle\langle\delta v_\varepsilon^2\rangle$$

$$+ 2\langle\delta v_\alpha^2\rangle\langle\delta v_\varepsilon^4\rangle + 2\langle\delta v_\alpha^2\rangle\langle\delta v_\varepsilon^2\rangle\langle\delta v_\gamma^2\rangle\Big]$$

$$= \delta_{\alpha\beta}\left[15\left(\frac{k_BT}{M}\right)^3 + 12\left(\frac{k_BT}{M}\right)^3 + 6\left(\frac{k_BT}{M}\right)^3 + 2\left(\frac{k_BT}{M}\right)^3\right]$$

$$= 35\left(\frac{k_BT}{M}\right)^3\delta_{\alpha\beta}$$

$$h_\alpha = -\frac{nM\tau_{MFP}}{2T}\left(\partial_{q_\beta}T\right)\left\langle\frac{M}{2kT}\left(\delta\vec{v}^2\right)^2\delta v_\beta\delta v_\alpha - \frac{5}{2}\delta\vec{v}^2\delta v_\alpha\delta v_\beta\right\rangle$$

$$= -\frac{nM\tau_{MFP}}{4T}\left(\partial_{q_\beta}T\right)\delta_{\alpha\beta}\left(\frac{M}{kT}\cdot 35\left(\frac{k_BT}{M}\right)^3 - 25\left(\frac{k_BT}{M}\right)^2\right)$$

$$\boxed{h_\alpha = -\frac{5}{2}\frac{n}{M}\tau_{MFP}k_B^2T\left(\partial_{q_\alpha}T\right)}$$

First-order hydrodynamics

$$D_t n = -n\,\partial_{q_\alpha}u_\alpha$$

$$M D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta}\cdot P_{\alpha\beta} = -\frac{1}{n}\partial_{q_\alpha}\left(nk_BT + \frac{2}{3}nk_BT\tau_{MFP}\partial_\gamma u_\gamma\right)$$

$$+\frac{1}{n}\partial_{q_\beta}\left[nk_BT\tau_{MFP}u_{\alpha\beta}\right]$$

$$D_tT = \frac{5}{3nk_B}\partial_{q_\alpha}\left[\frac{n}{M}\tau_{MFP}k_B^2T\,\partial_{q_\alpha}T\right] - \frac{2T}{3}\left[\partial_\alpha u_\alpha - \tau_{MFP}u_{\alpha\beta}u_{\alpha\beta} + \frac{2}{3}\tau_{MFP}(\partial_\alpha u_\alpha)^2\right]$$

Relaxation to equilibrium

$n = n_0 + \delta n$; $u = \delta u$; $T = T_0 + \delta T$

$\Rightarrow$ linearized dynamics for $\begin{pmatrix}\delta n\\ \delta u\\ \delta T\end{pmatrix}$: $\partial_t\begin{pmatrix}\delta n\\ \delta u\\ \delta T\end{pmatrix} = M\cdot\begin{pmatrix}\delta n\\ \delta u\\ \delta T\end{pmatrix}$

The eigenvalues of M have negative real parts, leading to the decay of the perturbation and the convergence to n & T uniform and $\vec{u} = \vec{0}$ $\Rightarrow$ global equilibrium.